JOURNAL OF ALGEBRA 5, 367-372 (1967)

Some Structure Theorems for Infinite Abelian P-Groups

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Received April 15, 1966

In what follows, the word "group" will mean "primary Abelian group" and unless otherwise specified the notation and terminology will be that of L. Fuchs in [2]. One exception to this will be that a direct sum of groups A and B will be denoted by $A \oplus B$ and $A + B$ will indicate a sum which is not necessarily direct.

We will also use the following definitions and theorems:

DEFINITION A (Irwin [3]). *Let G be a group. If H is a subgroup of G maximal with respect to disjointness from the elements of infinite height of G, then H will be called a high subgroup of G.*

THEOREM B (Irwin [3]). *Let G be a group. If H is a high subgroup of G, then H is pure in G and H contains a basic subgroup of G.*

DEFINITION C (Irwin and Walker [4]). *Let G be a group such that every high subgroup of G is a direct sum of cyclic groups, then G will be called a Σ-group.*

THEOREM D (Irwin and Walker [4]). *Let G be a group and H a high subgroup of G. If H is a direct sum of cyclic groups, G is a Σ-group.*

One should also recall that if G is a group then G^1 denotes the elements of infinite height in G.

Problem. Let G be a p-group and B a basic subgroup of G. If $r(G/B) < |G|$ then is this always due to the fact that a "large summand" of G is a direct sum of cyclic groups.

This problem is answered, when $G^1 = 0$, by Theorem 33.4 in [2]. This problem is answered, in general, as a consequence of Theorem 1. Some additional implications of that theorem are then given.

* This paper was partially supported by NSF Grant GP-377.

THEOREM 1. *Let G be an infinite reduced p-group and B a basic subgroup of G such that $G/B = \sum_{\alpha \in I} (G_\alpha/B)$ where $G_\alpha/B \simeq Z(p^\infty)$ for all $\alpha \in I$. Then $G = H \oplus K$ and $B = H \oplus L$ where L is a basic subgroup of K such that $r(K/L) = r(G/B) = |I|$ and $|K| =$ maximum $\{\aleph_0, |I|\}$.*

Proof. If $G/B = 0$ then the theorem is obvious, thus assume I is nonvoid. Let $\{x_i^\alpha + B\}_{i=1}^\infty$ be a system of generators of G_α/B such that for each $i = 1, 2, \ldots$ we have $px_{i+1}^\alpha + b_i^\alpha = x_i^\alpha$ where $b_i^\alpha \in B$. Let

$$L' = \{b_i^\alpha \mid \alpha \in I, i = 1, 2, \ldots\}.$$

Since B is a direct sum of cyclic groups and L' cannot be finite then B can be split as $B = H \oplus L$ where $L' \subset L$ and $|L'| = |L|$. Now $G/L = K/L \oplus R/L$ where K/L is the divisible subgroup of G/L and R/L is a reduced complementary summand chosen to contain B/L. Now $L' \subset L$ so $px_{i+1}^\alpha + L = x_i^\alpha + L$, hence for fixed $\alpha \in I$, $\langle\{x_i^\alpha + L \mid i = 1, 2, \ldots\}\rangle$ is divisible. Therefore $\langle\{x_i^\alpha + L \mid \alpha \in I, i = 1, 2, \ldots\}\rangle$ is divisible and

$$\langle\{x_i^\alpha \mid \alpha \in I, i = 1, 2, \ldots\}, L\rangle \subset K.$$

Recall $G = \langle\{x_i^\alpha \mid \alpha \in I, i = 1, 2, \ldots\}, B\rangle = \langle\{x_i^\alpha \mid \alpha \in I, i = 1, 2, \ldots\}, H, L\rangle$. Thus $\langle H, K\rangle = G$. Since $K \cap R = L$ we have $K \cap H = 0$ hence $G = H \oplus K$.

Now L as a summand of B is a pure direct sum of cyclic groups and $K/L \simeq (H \oplus K)/(H \oplus L) = G/B$ which is divisible, so L is a basic subgroup of K.

As before $G/B = (H \oplus K)/(H \oplus L) \simeq K/L$ so $|I| = r(G/B) = r(K/L)$. By the method of choosing L' and L we have

$$\aleph_0 \leqslant |L'| = |L| \leqslant \text{maximum}\{\aleph_0, |I|\},$$

and since $r(K/L) = |I|$ we also have $|K| = |L|\,|I|$. Thus

$$|K| = |L|\,|I| \leqslant (|I|)(\text{maximum}\{\aleph_0, |I|\}) = \text{maximum}\{\aleph_0, |I|\}$$

and $|K| = |L|\,|I| \geqslant |I|\,\aleph_0 = \text{maximum}\{\aleph_0, |I|\}$ so

$$|K| = \text{maximum}\{\aleph_0, |I|\}.$$

LEMMA 2. *Let G be a infinite reduced p-group and B an upper basic subgroup of G. If $r(G/B) = |G|$ then every basic subgroup of G is an upper basic subgroup of G and also a lower basic subgroup of G.*

Proof. Let A be an arbitrary basic subgroup of G and let L be a lower basic subgroup of G. Then $|G| \geqslant r(G/L) \geqslant r(G/A) \geqslant r(G/B) = |G|$. Therefore, G has the desired property.

The next theorem is a generalization of Theorem 33.4, p. 113 in [2].

THEOREM 3. *Let G be an infinite p-group such that $G^1 = 0$ or $|G^1| \geqslant \aleph_0$. Then $G = H \oplus K$ where H is an direct sum of cyclic groups and every basic subgroup of K is both an upper basic subgroup of K and a lower basic subgroup of K.*

Proof. Let B be an upper basic subgroup of G. Then there are two possible cases.

Case (1). If $r(G/B) = |G|$, then by Lemma 2, we can let $G = K$ and $H = 0$ and both H and K will have the desired properties.

Case (2). If $r(G/B) < |G|$, then by Theorem 1, $G = H \oplus K$ and $B = H \oplus L$ where H is a direct sum of cyclic groups and L is a basic subgroup of K. Now it is clear that L is an upper basic subgroup of K because if it is not then B would not be an upper basic subgroup of G. Now by Theorem 1 $|K| = \text{maximum}\{\aleph_0, r(G/B)\}$. If $r(G/B) < \aleph_0$ then G^1 must be finite; hence by assumption $G^1 = 0$. If this happens, then since $|K| = \aleph_0$ and $K^1 = 0$, K is a direct sum of cyclic groups and so G is a direct sum of cyclic groups; therefore if one lets $G = H'$ and $K' = 0$ then the theorem holds for $G = H' \oplus K'$.

Now if $r(G/B) \geqslant \aleph_0$, then $|K| = \text{maximum}\{\aleph_0, r(G/B)\} = r(G/B) = r(K/L)$, so that by Lemma 2 every basic subgroup of K is an upper basic subgroup of K and a lower basic subgroup of K. Therefore, $G = H \oplus K$ where H and K have the desired properties.

One should observe at this point that if $0 < |G^1| < \aleph_0$ the theorem does not always hold. To see this one only needs to consider a countable group G with $0 < |G^1| < \aleph_0$.

The next theorem is a slight generalization of Theorem 4, p. 1547 in [*1*].

THEOREM 4. *Let G be an infinite reduced p-group such that $|G^1| \leqslant \aleph_0$ and G is a Σ-group. Then G is a direct sum of countable groups.*

Proof. Let B be a high subgroup of G. Since G is a Σ-group then B is basic subgroup of G (by Theorem B and by Theorem 5c, p. 1380, in [*3*]) and we have $r(G/B) = r(G^1[p]) \leqslant \aleph_0$. By Theorem 1, $G = H \oplus K$ where H is a direct sum of cyclic groups and $|K| = \text{maximum}\{\aleph_0, r(G/B)\} = \aleph_0$. Therefore, G has the desired properties.

COROLLARY 5. *Two infinite reduced p-groups that are Σ-groups each with at most a countable number of elements of infinite height are isomorphic if and only if they have the same Ulm invariants.*

Proof. Since Kolettis [*5*] has proven that two reduced p-groups (which are both direct sums of countable groups) are isomorphic if and only if they have the same Ulm invariants, then an application of Theorem 4 makes this corollary obvious.

Theorem 9 is a generalization of Theorem 4 and actually suggests a generalization of Ulm's Theorem that will appear in another paper at a latter date. However, we must first prove the following lemmas and Theorem 8 which are needed in the proof of Theorem 9.

LEMMA 6. *Let G be a p-group and B a basic subgroup of G. If N is a neat subgroup containing B, then N is pure.*

Proof. Since N is neat in G then N/B is neat in G/B. However, G/B is divisible and any neat subgroup of a divisible group is pure, so that N/B is pure in G/B. Now since B is pure in G then N is also pure in G.

LEMMA 7. *Let G be a p-group, B a basic subgroup of G, and N be a neat subgroup of G^1. If M is a minimal neat subgroup of G containing B and N, then M is pure in G and $M^1 = N$.*

Proof. By Lemma 6, M is a pure subgroup of G so we have $N \subset M^1$. Since N is a neat subgroup of G^1 it is also a neat subgroup of M^1 so to show $N = M^1$ it suffices to show $N[p] = M^1[p]$. Now

$$M[p] = (B[p]) \oplus (N[p])$$

and it is clear that

$$M[p] \supset (B[p]) \oplus (M^1[p]) \supset (B[p]) \oplus (N[p]) = M[p].$$

Therefore, since $N[p] \subset M^1[p]$ it is clear that $M^1[p] = N[p]$.

THEOREM 8. *Let G be a p-group, let K be a high subgroup of G and let B be a basic subgroup of G^1. If N is a minimal neat subgroup of G such that $N \supset B \oplus K$ then $G/B = (G^1/B) \oplus (N/B)$.*

Proof. Since K is a high subgroup of G, K is pure and contains a basic subgroup of G, by Theorem B. Since B is a basic subgroup of G^1 it is neat in G^1; so by Lemma 7, N is a pure subgroup of G and $N^1 = B$. Therefore, $(N/B)^1 = 0 + B$. Now G^1/B is divisible so that $G/B = (G^1/B) \oplus (R/B)$ where R/B can be chosen such that $N/B \subset R/B$. Now $(G/B)/(N/B) \cong G/N$ is divisible and

$$G/N \cong (G/B)/(N/B) \cong (G^1/B) \oplus ((R/B)/(N/B)) \cong (G^1/B) \oplus (R/N).$$

Thus R/N is divisible; hence R/N is a pure subgroup of G/N. However, N is a pure subgroup of G so R is a pure subgroup of G. Now

$$R/B \cong (G/B)/(G^1/B) \cong G/G^1$$

so $(R/B)^1 = 0 + B$. Therefore, $R^1 = B$. Now if R properly contains N then since $R^1 = N^1$, R must have a high subgroup L which properly contains K. However, this is impossible because then L would be high in G. Therefore, $R = N$ and $G/B = (G^1/B) \oplus (N/B)$.

THEOREM 9. *Let G be an infinite reduced p-group such that G is a Σ-group and the cardinality of a basic subgroup of G^1 is at most countable. Then $G = H \oplus K$ where H is a direct sum of cyclic groups and the cardinality of a high subgroup of K is at most countable.*

Proof. Let L be a high subgroup of G and let B be a basic subgroup of G^1. Let N be a minimal neat subgroup of G such that $B \oplus L \subset N$. Then by Theorem 8, $G/B = (G^1/B) \oplus (N/B)$ and, by Lemma 7, $N^1 = B$. Now by Theorem 4 we have $N = R \oplus S$ where S is a direct sum of cyclic groups and $N^1 = R^1 = B$ and $| R | \leqslant \aleph_0$. So

$$G/B = (G^1/B) \oplus (N/B) = (G^1/B) \oplus (R/B) \oplus ((B \oplus S)/B)$$

and therefore, $G = (G^1 + R) \oplus S$. By Lemma 7, N is pure in G so R is pure in G. Let A be a basic subgroup of R. Then A is pure in G and hence A is pure in $G^1 + R$. Also, since $(G^1 + R)/A$ is clearly divisible and A is a direct sum of cyclic groups, then A is a basic subgroup of $G^1 + R$. Let T be a high subgroup of $G^1 + R$ such that $A \subset T$; then $T \oplus S$ is clearly a high subgroup of G. Now G is a Σ-group so T is a direct sum of cyclic groups. However, since A is a basic subgroup of T then $A \cong T$. Now $| R | \leqslant \aleph_0$ so $| A | \leqslant \aleph_0$ so that $| T | = | A | \leqslant \aleph_0$. Therefore, if we let $H = S$ and $K = G^1 + R$, then $G = H \oplus K$ where H and K have the desired properties.

The proof of Theorem 9 suggests the following theorem.

THEOREM 10. *Let G be an infinite reduced p-group such that G is Σ-group and the cardinality of a basic subgroup of G^1 is at most countable. Let B be a basic subgroup of G. Then $G/G^1 \cong B$.*

Proof. By Theorem 9 we have $G = H \oplus K$ where H is a direct sum of cyclic groups and the cardinality of a high subgroup of K is at most countable. Let L be a high subgroup of K then $H \oplus L$ is a basic subgroup of G. By Lemma 3 in [*3*], we have $K^1 = G^1$. Let A be a basic subgroup of K^1, and let N be a minimal neat subgroup of K containing $L \oplus A$. Observe $| N | \leqslant \aleph_0$ and by Lemma 7 we have that N is a pure subgroup of K with $N^1 = A$ and that L is a basic subgroup of N. Now by Theorem 8 we have $K/A = (K^1/A) \oplus (N/A)$ so that $K/K^1 \cong (K/A)/(K^1/A) \cong N/A$. By Theorem 30.2, p. 103 in [*2*], we know $(L \oplus A)/A$ is a basic subgroup of N/A. Since $| N/A | \leqslant \aleph_0$ and $(N/A)^1 = 0 + A$ we see that N/A is a direct sum of cyclic groups and thus $N/A \cong (L \oplus A)/A \cong L$. Therefore

$$\begin{aligned} B &\cong L \oplus H \cong (N/A) \oplus H \cong (K/K^1) \oplus H \cong (K/G^1) \oplus H \\ &\cong (H \oplus K)/G^1 = G/G^1. \end{aligned}$$

ACKNOWLEDGMENT

The authors wish to express their gratitude to Professor John M. Irwin for his advice in preparing this paper.

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